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AMALGAMATION BASES FOR COMMUTATIVE RINGS WITHOUT NILPOTENT ELEMENTS

BY

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ABSTRACT

SCR = the class of commutative rings without nilpotent elements.

THEOREM. R is an amalgamation base for SCR iff $\mathrm{rad}(I) = \mathrm{Ann}^2(I)$ for $I \subseteq R$ finitely generated.

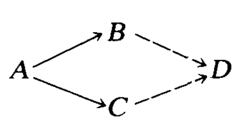
SUPPLEMENT. If $R \in \mathrm{SCR}$ then R is contained in an amalgamation basis for SCR having no new idempotent elements.

CR = the class of commutative rings.

THEOREM. R is an amalgamation base for CR iff R is a pure R-submodule of any commutative ring extending R.

Introduction

Let K be a class of structures. An element A of K is said to be an *amalgamation base* for K iff for any two structures $B, C \subseteq K$ which contain A, there is a structure D containing B and C. In diagrammatic form:



Our main result is a precise description of the amalgamation bases for the class SCR of semiprime commutative rings (commutative rings without nilpotents). We will also characterize the amalgamation bases for CR (commutative rings), but the result is less satisfactory. We note that all rings with which we deal are assumed to have an identity.

This work was motivated by a result in an anonymous letter received by Weispfenning, stating that the class of amalgamation bases for SCR is not axiomatizable.

§1. Amalgamation bases for SCR

DEFINITION 1.1. Let I be an ideal of the commutative ring A, $\operatorname{rad} I = \{a \in A : \text{for some } n\ a^n \in I\}$. By abuse of notation, $\operatorname{rad}(0)$ is also denoted $\operatorname{rad} A$. The annihilator of I ($\operatorname{Ann} I$) is defined to be $\{a \in A : aI = 0\}$. We write $\operatorname{Ann}^2(I)$ for $\operatorname{Ann}(\operatorname{Ann} I)$.

A is said to be semiprime iff $\operatorname{rad} A = (0)$. A is said to be regular iff for every $a \in A$ there is an x such that

$$a^2 x = a.$$

We assume some familiarity with regular rings.

THEOREM 1.2. *The semiprime ring A is an amalgamation base for SCR iff for each finitely generated ideal I of A we have*:

$$\operatorname{Ann}^2(I) = \operatorname{rad} I. \tag{1}$$

REMARK 1.3. In any semiprime ring A we always have $\operatorname{rad} I \subseteq \operatorname{Ann}^2(I)$. (If $a \in \operatorname{rad} I$ and $b \in \operatorname{Ann} I$, then for a large n, $a^n b = 0$, hence $(ab)^n = 0$ and $ab = 0$.) Furthermore if A is regular we will always have (1), as follows: if $a \in \operatorname{Ann}^2(I)$ where $I = (b_1, \cdots, b_k)$ let e be the union of the idempotents e_i associated to the b_i by the formula $e_i = b_i x_i$ where $b_i^2 x_i = b_i$. Cf. [2]. Then $e \in I$ and $1 - e \in \operatorname{Ann} I$. Thus $a(1-e) = 0$, $a = ea$, and $a \in I$, hence $a \in \operatorname{rad} I$. Thus regular rings are amalgamation bases for SCR, as is known.

After proving Theorem 1.2 we will give further examples of (nonregular) amalgamation bases. For convenience we divide the proof into two lemmas.

LEMMA 1.4. *Let A be a commutative semiprime ring, $I = (a_1, \cdots, a_k)$ a finitely generated ideal of A, $b \in \operatorname{Ann}^2(I)$, $b \in \operatorname{rad}(I)$. Then there are semiprime extensions B, C of A such that*:

1) $b \notin \operatorname{Ann}_B^2(IB)$,

2) $b \in IC$.

PROOF. 1) Let $B' = A[x]/(xI)$, $B = B'/\operatorname{rad} B'$. Then the canonical homomorphism $A \to B$ is injective. Indeed the map $x \to 0$ induces a commutative diagram

$$\begin{array}{ccc} A & \rightarrow & B' \\ & \diagdown\!\!\diagdown \quad \swarrow & \\ & A & \end{array},$$

so the map $A \to B'$ is injective. On the other hand, if $a \in A$ and $a = 0$ in B, then for some n, $a^n = 0$ in B', so $a^n = 0$ in A, and hence $a = 0$ in A, as desired. Thus B extends A, and in B the element x annihilates I.

On the other hand $bx \neq 0$ in B, for otherwise we would have a relation in B':

$$b^n x^n = \sum_j x i_j p_j(x).$$

Equating terms of degree n, we find $b^n \in I$, so $b \in \mathrm{rad} I$, a contradiction.

2) Let $C' = A[x_1, \cdots, x_k]/(\Sigma a_i x_i - b)$. Let $C = C'/\mathrm{rad} C'$. We claim that C' (and hence C) extends A. As we know that $b \in IC$, this will complete the proof.

The verification that C' extends A consists of a rather long, essentially trivial computation. Assume that there is a relation:

(2) $$a = p(\bar{x})(\Sigma a_i x_i - b) \quad \text{in} \quad A[\bar{x}].$$

We claim that $a = 0$. In fact we claim that b annihilates all coefficients of $p(\bar{x})$.

For notational convenience use multi-indices, writing:

$$p(\bar{x}) = \sum_J p_J x^J$$

where $J = (j_1, \cdots, j_k)$, $p_J \in A$ for all J, $p_J = 0$ for all but finitely many J, and x^J is (by definition)

$$x_1^{j_1} \cdots x_k^{j_k}.$$

We will prove by downward induction on $|J| = \Sigma_i j_i$ that $bp_J = 0$. It then follows that $a = 0$. If $|J|$ is large then $p_J = 0$, so $bp_J = 0$. Assume therefore that n is fixed, and $bp_J = 0$ whenever $|J| = n + 1$. We propose to show that $bp_J = 0$ whenever $|J| = n$.

This will be proved by induction over a variant of the lexicographic ordering. For σ a permutation on $\{1, \cdots, k\}$, let $J^\sigma = (j_{1^\sigma}, \cdots, j_{k^\sigma})$. Say that J_1 is equivalent to J_2 iff for some permutation σ, $J_1^\sigma = J_2^\sigma$. Say that J_1 *dominates* J_2 iff for some permutation $\sigma \in S_k$, J_1^σ is lexicographically larger than J_2. This relation induces a linear ordering of type ω^k on the equivalence classes. For any multi-index J and integers i $(1 \leqq i \leqq k)$ let $J/i = (j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_k)$. We will prove that if $|J| = n$ then for each i $(1 \leqq i \leqq k)$:

(3) $$a_i p_J = 0.$$

This will prove that each p_J annihilates $(a_1, \cdots, a_k)$, hence $bp_J = 0$ ($L \in \mathrm{Ann}^2(0)$). We prepare to prove (3) (for all suitable i, J) by transfinite induction on the height of J/i with respect to the domination relation (the relevant permutation group here is S_{k-1}). To start the induction we claim:

(4) $$\text{If } J/i = (0, \cdots, 0) \text{ then } a_i p_J = 0.$$

We must write out the relation (2) explicitly; we use the notation $\delta_i = (\delta_{i0}, \cdots, \delta_{ik})$ (the Kronecker delta). For each J, (2) yields:

$$bp_{J+\delta_i} = \sum_{(j_r+\delta_{ir}>0)} a_r p_{J+\delta_i-\delta_r}. \tag{5}$$

If $|J| = n$, then by hypothesis $bp_{J+\delta_i} = 0$. In particular if $J/i = (0, \cdots, 0)$ we get:

$$a_i p_J = 0$$

as desired.

To proceed with the induction, assume that $|J| = n$, $1 \leqq i \leqq k$, and $a_{i'} p_{J'} = 0$ whenever J'/i' is dominated by J/i. We will show that $a_i p_J = 0$. We examine (5):

$$0 = \sum_{\substack{r \\ (j_r+\delta_{ir})>0}} a_r p_{J+\delta_i-\delta_r}. \tag{6}$$

It suffices to show that each term $a_r p_{J+\delta_i-\delta_r} = 0$ in (6). Call r *refractory* if $a_r p_{J+\delta_i-\delta_r} \neq 0$, and rewrite (6) as:

$$0 = \sum_{r \text{ refractory}} a_r p_{J+\delta_i-\delta_r}. \tag{7}$$

Choose r refractory so that j_r is minimal.

Multiply (7) by $p_{J+\delta_i-\delta_r}$, obtaining:

$$0 = \sum_{s \text{ refractory}} a_s p_{J+\delta_i-\delta_s} p_{J+\delta_i-\delta_r}. \tag{8}$$

Now for $s \neq r$, $J + \delta_i - \delta_r/s$ is dominated by $J + \delta_i - \delta_r/r$. It follows that for $s \neq r$, $a_s p_{J+\delta_i-\delta_r} = 0$, and (8) reduces to:

$$0 = a_r p^2_{J+\delta_i-\delta_r}. \tag{9}$$

Since A is semiprime,

$$a_r p_{J+\delta_i-\delta_r} = 0, \tag{10}$$

contradicting the assumption that r is refractory.

EXAMPLE 1.5. Let $A' = F[a_1, a_2, b, r_1, r_2, s]$ be a polynomial ring in six indeterminates over a field F. Let J be the ideal of A' generated by:

1) All products of three indeterminates.
2) All squares of indeterminates.
3) $a_1a_2, a_1b, a_2b, r_1r_2, r_1s, r_2s, a_1r_1, a_2r_2$.
4) $a_1r_2 + a_2r_1, a_1s - br_1, a_2s - br_2$.

Let $A = A'/J$. Then A has an F-basis consisting of

$$1, a_1, a_2, b, r_1, r_2, s, a_1r_2, a_1s, a_2s, bs$$

$\mathrm{Ann}(a_1, a_2) = (a_1, a_2, b)$ and $\mathrm{Ann}^2(a_1, a_2) = (a_1, a_2, b)$. In particular $b \in \mathrm{Ann}^2(a_1, a_2)$. On the other hand, the polynomial identity in $A[u, v]$:

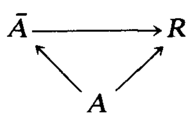
$$bs = (b - a_1u - a_2v)(s + r_1u + r_2v)$$

shows that $b \in (a_1, a_2)$ in any extension of A.

PROOF OF THEOREM 1.2 (Necessity).

It is now clear that in every semiprime amalgamation base we have $\mathrm{Ann}^2(I) = \mathrm{rad} I$ for finitely generated ideals. Indeed, if I is finitely generated and $b \in \mathrm{Ann}^2(I) - \mathrm{rad}(I)$, we may construct according to Lemma 1.4 two semiprime extensions B, C of A which evidently cannot be amalgamated (if $B, C \subseteq D$, then one would have $b \in ID$ and $b \notin \mathrm{Ann}^2_D(ID)$, which is nonsense).

In the proof of the second half of Theorem 1.2 we will use the *regular hull* of a semiprime ring A, that is the ring $\bar{A}$ generated by A and symbols $\{a^{-1}: a \in A\}$ subject to the relations of A and the relations $\{a^2a^{-1} = a,\ a(a^{-1})^2 = a^{-1}\}$ (compare [5, 3]). This is characterized by the existence of a unique homomorphism ϕ such that:

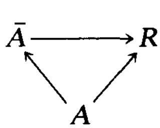


commutes

whenever R is a regular extension of A. We will assume a certain acquaintance with idempotents in regular rings (cf. [2]).

LEMMA 1.6. *Let A be semiprime, and suppose that* $\mathrm{Ann}^2(I) = \mathrm{rad}(I)$ *for all finitely generated ideals I of A. Let R be a regular extension of A generated by A (under $+$, $-$, $\cdot$, $^{-1}$). Then $R \simeq \bar{A}$, the regular hull of A, under the canonical induced homomorphism $\bar{A} \to R$.*

PROOF. The canonical induced homomorphism



will of course be surjective in the present instance, so we must verify that the kernel is trivial. Suppose then that we have in R:

(11) $$p(a_1,\cdots,a_l, \qquad a_1^{-1},\cdots,a_l^{-1})=0$$

for certain elements $\bar{a}$, $\bar{a}^{-1}$ of R. We must show that (11) is a consequence of the relations in A and the relations: $a^2a^{-1}=a$, $a(a^{-1})^2=a^{-1}$. It will be convenient to make use of the idempotents $e_i = a_ia_i^{-1}$, and of Boolean combination of these idempotents.

Define a *bit* of the $\{e_i\}$ to be a product of $B = \Pi_1^d f_i$ where each f_i is either e_i or its complement $(1-e_i)$.

Evidently $p(\bar{a},\bar{a}^{-1})=0$ iff for each bit f

(12) $$fp(\bar{a},\bar{a}^{-1})=0.$$

Define $f_1=\Pi\{f_i : f_i = e_i\}$, $f_2=\{\Pi f_i : f_i = 1-e_i\}$, so that $f=f_1f_2$. Let $a=\{\Pi a_i : f_i = e_i\}$, and let N be a large integer. Then (12) is equivalent to each of the following:

$$f_1f_2p(\bar{a},\bar{a}^{-1})=0$$

$$a^Nf_1f_2p(\bar{a},\bar{a}^{-1})=0$$

$$f_2a^Np(\bar{a},\bar{a}^{-1})=0.$$

Taking N large and replacing $p(\bar{a},\bar{a}^{-1})$ by $a^Np(\bar{a},\bar{a}^{-1})$ we may assume that no monomials including a_i^{-1} occur in p when $f_i=e_i$. On the other hand, since $f_2a_i^{-1}=0$ when $f_i=1-e_i$, one may assume that p is a function of the $\bar{a}$ alone. Thus (12) becomes:

(13) $$f_2p(\bar{a})=0.$$

Let $b=p(\bar{a})$. We may number the a_i so that for $1\leqq i\leqq k$ $f_i=1-e_i$ and for $k+1\leqq i\leqq l$ $f_i=e_i$. To conclude our argument we prove that each of the following statements implies the next (in fact they are evidently equivalent):

1) $f_2p(\bar{a})=0$ in R.

2) $b\in \mathrm{Ann}^2(a_1,\cdots,a_k)$ in A.

3) $b\in \mathrm{rad}(a_1,\cdots,a_k)$ in A.

4) $f_2p(\bar{a})=0$ in A.

($1\Rightarrow 2$): If $c\in\mathrm{Ann}(a_1,\cdots,a_k)$ then for $1\leqq i\leqq k$ $ce_i=0$, i.e. $cf_i=c$. Thus $c=f_2c$, and then $bc=f_2bc=0$.

($2\Rightarrow 3$): By assumption on A.

($3\Rightarrow 4$): Since $b\in\mathrm{rad}(a_1,\cdots,a_k)$ in A, the same statement holds *a fortiori* in $\bar{A}$. As we noted earlier, it follows that $b\in\mathrm{Ann}^2(a_1,\cdots,a_k)$ (in $\bar{A}$). Since $f_2\in\mathrm{Ann}(a_1,\cdots,a_k)$, we conclude that $f_2b=0$.

PROOF OF THEOREM 1.2 (Sufficiency).

It is known that regular commutative rings are amalgamation bases for SCR (cf. [4]).

If we are given any semiprime ring A in which radicals and double annihilators of finitely generated ideals coincide, consider any diagram

$$\begin{array}{ccc} & & B \\ & \nearrow & \\ A & & \\ & \searrow & \\ & & C \end{array}$$

of semiprime rings and embeddings. We may take B, C to be regular without loss of generality. In particular each element a has an "inverse" a^{-1} in B determined by

$$a^2a^{-1} = a, \qquad a(a^{-1})^2 = a^{-1};$$

and similarly a has an "inverse" in C. Let A_1 be the subring of B generated by $A \cup \{a^{-1}: a \in A\}$, and let A_2 be the corresponding ring of C. We remark that A_1, A_2 are regular subrings of B, C.

By Lemma 1.6 $A_1 \simeq A_2$ over A. Identifying A_1, A_2, consider the diagram:

$$\begin{array}{ccccc} & & & & B \\ & & & \nearrow & \\ A & \longrightarrow & A_1 & & ; \\ & & & \searrow & \\ & & & & C \end{array}$$

this can be amalgamated over A_1, hence over A.

Everything proved so far leaves open the possibility that all amalgamation bases for SCR are in fact regular. To ensure a supply of nonregular amalgamation bases we prove:

THEOREM 1.7. *Let A be a semiprime commutative ring. Then there is an extension $B \supseteq A$ having the same idempotents as A, such that B is an amalgamation base for SCR.*

PROOF. It suffices to show that for any finitely generated ideal I and any $a \in A - \mathrm{rad}(I)$, it is possible to find an extension B of A having the same idempotents as A, in which $a \notin \mathrm{Ann}^2(IB)$. To this end we set $B = A[x]/\mathrm{rad}(xI)$. We know that B extends A, $x \in \mathrm{Ann}(I)$, and $ax \neq 0$ (Lemma 1.4(1)).

To conclude our argument, we must show that for any $p(x) \in A[x]$ such that $(p^2 - p) \in \mathrm{rad}(xI)$, there is an idempotent $e \in A$ such that $(p - e) \in \mathrm{rad}(xI)$.

Assume then that $p^2 - p \in \mathrm{rad}(xI)$; it follows easily that the constant term

$e = p(0)$ of p is idempotent. Write $p(x) = e + xq(x)$. Our claim is that $xq \in \text{rad}\, xI$. We have:

$$p^2 - p = [(2e-1) + xq(x)]xq(x) \in \text{rad}(xI). \tag{14}$$

If $q(x) = \Sigma a_i x^i$ our claim is that each $q_i \in \text{rad}(I)$. Indeed, in the contrary case write $q = q_0(x) + x^m q_1(x)$ where $q_0(x) \in (\text{rad} I)$ and the constant term $q_1(0)$ of q_1 is not in rad$I$. (14) yields:

$$[(2e-1) + x^{m+1}q_1(x)]x^{m+1}q_1(x) \in \text{rad}(xI). \tag{15}$$

Since $(2e-1)$ is invertible (in fact $(2e-1)^2 = 1$) it follows that $q_1(0) \in \text{rad}(xI)$, a contradiction.

The simplest way to obtain nonregular amalgamation bases is by starting with a ring A whose only idempotents are 0, 1, but which is an integral domain. The extension of A afforded by Theorem 1.7 cannot be regular.

Theorem 1.2 implies that the class of amalgamation bases is not axiomatizable, a result cited in the introduction. More precisely:

THEOREM 1.8. *Let K be a class of amalgamation bases for SCR, and assume K is axiomatizable (or just: closed under ultrapowers). Then K is contained in the class of regular commutative rings.*

PROOF. Suppose on the contrary that $A \in K$ is not regular, so that we have an element $a \in A$ such that $a \notin (a^2)$. It then follows that $a^n \notin (a^{n+1})$ for each n, for if $a^n = a^{n+1}x$ then $(a - a^2x)^n = a^n(1-ax)^n = 0$, so $a = a^2x$, and $a \in (a^2)$.

Take a nonprincipal ultrapower A^* of A over ω, and let b, $c \in A^*$be represented by the functions $(a, a, a, \cdots)$ and $(a, a^2, a^3, \cdots)$. Then it is easily seen that $b \in \text{Ann}^2(c)$ but $b \notin \text{rad}(c)$, contradicting the assumption that A^* is an amalgamation base.

§2. Amalgamation bases for CR

Let CR be the class of commutative rings. Recall that a module M over a ring A is a pure submodule of the A-module M' iff for any other A-module N the sequence:

$$0 \to M \otimes N \to M' \otimes N \tag{16}$$

is exact.

An equivalent condition is that for any $l \times k$ matrix T over A and elements $m_1, \cdots, m_k$ of M, the system of equations in indeterminates $x_1, \cdots, x_l$:

(17) $$\bar{x}T = \bar{m}$$

is solvable in M iff it is solvable in M'.

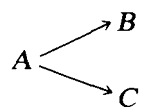
Yet another formulation of purity is obtained using:

FACT 1.9 (see [1]). The system of equations (17) is solvable in an extension of M iff for each $\bar{u} \in A^k$ such that $T\bar{u} = \bar{0}$, we have $\bar{m} \cdot \bar{u} = 0$.

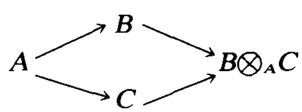
The following theorem is scarcely surprising.

THEOREM 1.10. *The ring A is an amalgamation base for CR iff for every commutative ring B extending A, A is a pure A-submodule of E.*

PROOF. Assume A is a pure A-submodule of each of its ring extensions. Then any diagram



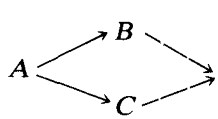
of commutative rings may be amalgamated as follows:



Next suppose that A is not a pure submodule of the ring $B \supseteq A$. Assume specifically that T is an $l \times k$ matrix over A, $\bar{r} \in A^k$ and the equations:

(18) $$\bar{x}T = \bar{r}$$

are solvable in B but not in A. We seek an extension C of A such that (18) is solvable in no extension of C. Then it is impossible to amalgamate the diagram:



and our argument will be complete.

We introduce indeterminates $\bar{u} = u_1, \cdots, u_k$ and let $C = A/(T\bar{u})$; in other words, if $T = (t_{ij})$ then $C = A/(\Sigma_j t_{ij} u_j : 1 \leqq i \leqq l)$. Of course C extends A, since the map $e_0 : \bar{u} \to \bar{0}$ induces a commutative diagram:

$$A \longrightarrow C \xrightarrow{e_0} A$$

Of course also $T\bar{u} = \bar{0}$ in C. We claim that $\bar{u} \cdot \bar{r} \neq 0$, so that (18) is insoluble in all extensions of C.

Suppose on the contrary that $\bar{u} \cdot \bar{r} = 0$ in C, so that we may write a polynomial identity in $A[\bar{u}]$:

$$\sum r_j u_j = \sum_i \sum_j t_{ij} u_j p_i(\bar{u}). \tag{19}$$

Let a_i be the constant term of $p_i(\bar{u})$, and equate the coefficients of u_j in (19):

$$r_j = \sum_i t_{ij} a_i. \tag{20}$$

Thus $\bar{a}T = \bar{r}$, which would solve (18) in A, a contradiction.

Theorem 1.10 would not appear to be the last word on the subject of amalgamation bases for CR, but it does provide useful information. In particular any ring A which is absolutely pure as an A-module is an amalgamation base. The following terminology may be found convenient.

Definition 1.11. Let A be a commutative ring. We will say that A is *module-pure* (respectively, *ring-pure*) iff A is a pure submodule of any A-module (respectively, commutative ring) containing A.

We have not yet excluded the possibility that ring-purity actually implies module-purity.

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